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Nonlinear Baroclinic Instability: An Approach Based on Serrin's Energy Method

PIERRE HENROTAY

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PIERRE HENROTAY

Mécanique des Fluides Géophysiques, Université de Liège, B6, 4000 Liège, Belgium

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ABSTRACT

The nonlinear problem of baroclinic instability in a two-layer fluid with Ekman dissipation is reformulated in the spirit of Serrin's method. Serrin's (1959) definition of stability in the mean and the adjoint variational problem are briefly presented, together with the improvements due to Joseph (1965). A linear combination of the total energy and the potential enstrophy appears to be a much better measure of the departure from the original motion than the conventional kinetic energy as in Serrin's original work.

1. Introduction

Numerous methods in nonlinear stability analysis are based on the use of positive definite functionals.

Among these methods are the Lyapunov (1907) direct method (which will not be discussed here) and the energy or Serrin's (1959) method, which is less popular in geophysical fluid dynamics. Interesting works by Arnold (1966), Blumen (1968) and others have drawn attention to the importance of a pertinent choice of these functionals. The aim of the present work is to show, in an illustrative example, how improved stability limits can be obtained by Serrin's variational method used with suitable functionals.

In Section 2, the well-known baroclinic stability problem in a two-layer fluid with Ekman friction is posed, and marginal stability curves are reminded. In Section 3, Serrin's variational method, together with the improvements due to Joseph (1965), is presented. In Section 4, the nonlinear stability problem is treated, first through the method described in the previous section (thus with energy as a positive functional), then with the potential enstrophy playing the role of the functional, and finally with a linear combination of both.

Stability curves obtained with each variational problem are compared; and an interesting result is that part of the linear stability curve is recovered by the nonlinear analysis.

2. The Reference problem—Linear stability results

The stability problem treated here as an application of our method is chosen to be simple enough to avoid useless lengthy calculations, but also to allow an interesting comparison with the linear stability results.

The problem is a well-known classic in the geophysical sciences and describes the joint effect of baroclinicity and Ekman dissipation in a two-layer model with equal depths in both layers (extension to the case with different ambient depths is straightforward). The upper and lower surfaces are rigid, the effect of the earth's curvature is neglected, and the system is gravitationally stable.

The flow is supposed to be periodic in the horizontal plane, but this assumption can easily be removed. In the quasi-geostrophic approximation, the primitive equations of motion can be reduced to the following set of nondimensional equations, which describe the horizontal motions in the region outside the Ekman friction layers [for a complete derivation, the reader is referred to Pedlosky (1970, 1979)]:

$$\left.\begin{aligned} \frac{\partial Q_1}{\partial t} + J(\psi_1, Q_1) &= -r\nabla^2\psi_1 \\ \frac{\partial Q_2}{\partial t} + J(\psi_2, Q_2) &= -r\nabla^2\psi_2 \end{aligned}\right\}, \qquad (2.1)$$

where ψ_1, ψ_2 are the geostrophic streamfunctions for the horizontal motion, $Q_i = \nabla^2\psi_i + \mathrm{F}(-1)^i(\psi_1 - \psi_2)$ (the potential vorticities), $J = \partial(\ ,\ )/\partial(x, y)$ is the Jacobian operator (x and y are space variables, typically west–east and south–north, ∇^2 is the horizontal Laplacian, and $r = \mathrm{E}^{1/2}/\epsilon$. (E, F and ϵ are respectively the vertical Ekman number, the internal Froude number and the Rossby number.)

The basic state whose stability is investigated consists of a vertically sheared zonal flow in which the velocities V_1 and V_2 in each layer are independent of y; the shear (assumed, without restriction, to be positive) will be noted as

$$V_S \equiv V_1 - V_2. \qquad (2.2)$$

The results of the linear stability analysis are simply quoted here from Pedlosky (1970, 1979): per-

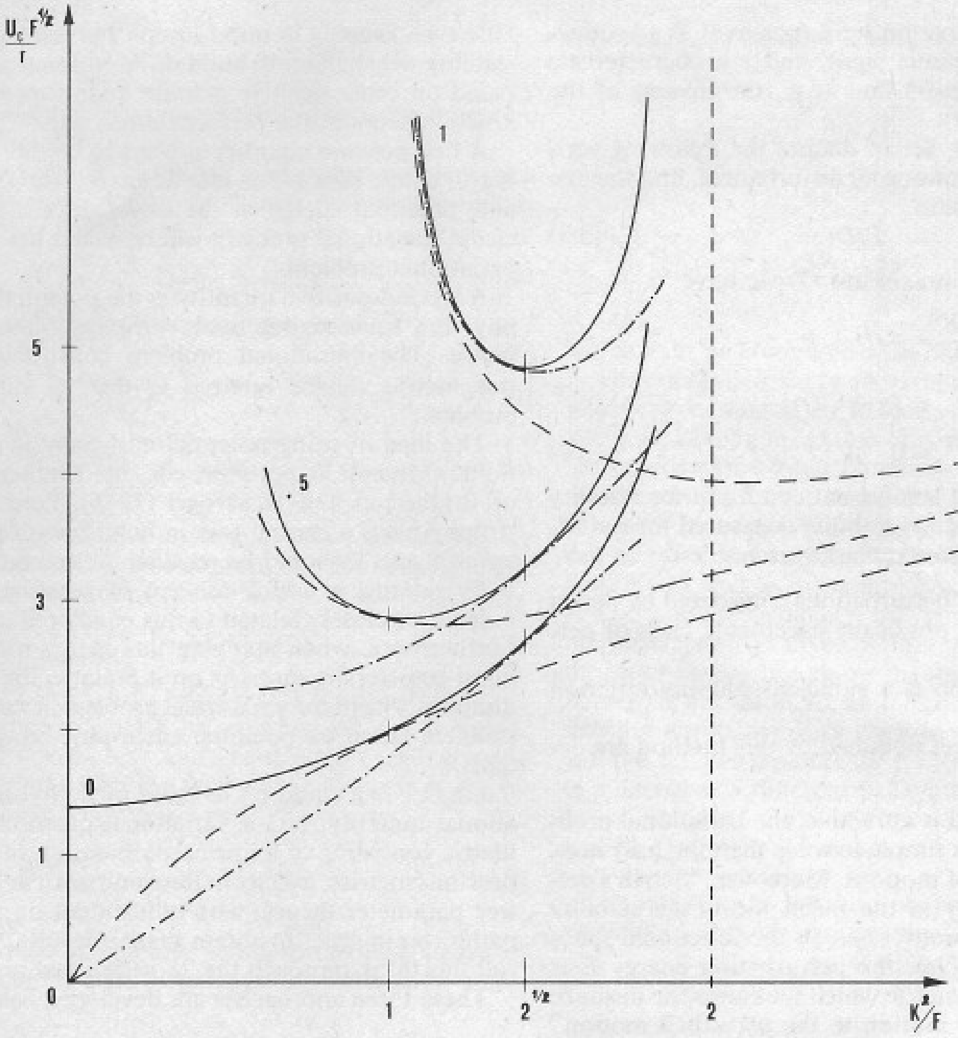

FIG. 1. Stability curves for various values of l^2/F: solid line, linear theory; dashed line, E-variational problem; dot-dashed line, Q-variational problem.

turbations $\phi_n(x, y, t)$ may be found in the form

$$\phi_n = \mathrm{Re} A_n \exp[i(kx + ly - \sigma t)], \quad n = 1, 2, \tag{2.3}$$

and a straightforward development gives the stability condition

$$V_S < V_C(k, K),$$

with

$$\frac{V_C \mathrm{F}^{1/2}}{r} = \frac{2K\mathrm{F}^{1/2}}{k(2\mathrm{F} - K^2)^{1/2}},$$

$$K^2 = k^2 + l^2. \tag{2.4}$$

Marginal stability is achieved for $V_S = V_C$ (Fig. 1, solid line).

An important remark must be formulated: the criterion is clearly valid for infinitesimal disturbances only and no information on the behavior of finite disturbances can be obtained. A basic flow, in general, may be stable against infinitesimals, and unstable against finite perturbations, but stability limits for these two cases need not be identical.

3. The Serrin and Serrin-Joseph variational methods

Serrin (1959) developed and extended the ideas of Orr (1907), based on the evolution of the global energy of the perturbation. Assuming the basic flow to be altered at some instant, this flow is said to be stable in the mean if the (space-integrated) kinetic energy of the perturbation tends to zero as time goes to infinity, i.e., if

$$\frac{dE}{dt} < 0. \tag{3.1}$$

The dimensionless equations of motion are used to derive the temporal variation of E and a resulting differential equation arises, of the form

$$\frac{dE}{dt} = -rD + P, \tag{3.2}$$

where D is a dissipation term (positive), P a production term (no definite sign), and r a characteristic number for the basic flow (e.g., the inverse of the Reynolds number).

To satisfy (3.1), Serrin defines the following variational problem: among all disturbances, find the one maximizing the ratio

$$P/D; \tag{3.3}$$

then, writing this maximum r^*, we have

$$\begin{aligned}\frac{dE}{dt} &= -rD + P \\ &\leqslant D(r^* - r) \\ &\leqslant 0 \quad \text{if} \quad r > r^*. \end{aligned} \tag{3.4}$$

In order to obtain a sufficient condition for stability $r > r^*$, while marginal stability is ensured for $r = r^*$.

At this stage, some remarks are needed.

1) The class of perturbations considered by Serrin is larger than the physically acceptable class of perturbations.

2) This criterion is a sufficient stability criterion only.

3) Critical values obtained by this method are default estimations.

Serrin's method is attractive: the variational problem is much more simple to solve than the fully nonlinear equations of motions. Moreover, "Serrin's definition of stability in the mean meets the stability concepts of Lyapunov. . . . In the functional space of the field variables, the perturbation energy E is indeed a suitable metric which measures the distance from the laminar motion to the perturbed motion" (Nihoul, 1969).

In order to obtain better estimates of the stability limit, Joseph (1965), in his study of the stability of a motionless fluid heated from below, proposed to consider the evolution of another positive definite quantity than the total kinetic energy of the perturbation, namely a linear combination of the kinetic and thermal energies. This linear combination allows the introduction of a free parameter λ, which can be selected in order to reach a better limit of stability. The great advantage is to point out that the kinetic energy is not always the most indicated metric for a stability problem (and indeed many contributions on nonlinear stability problems have been published, using more sophisticated metrics (e.g., Blumen, 1968), but mainly based on Lyapunov's method) and that other positive quantities can also provide a start to a variational problem of Serrin's type. Our work must be replaced within this context.

4. Nonlinear stability analysis

Our task here will be to derive sufficient conditions for stability, following Serrin's variational approach. However, keeping in mind Joseph's investigations on stability, we shall try to build up variational problems based on other positive definite quantities than the kinetic energy of the perturbation.

A first positive quantity appears to be the total energy, i.e., the sum of the kinetic energy and the available potential energy of the disturbance. The associated variational problem will be called here the "E variational problem."

A second positive quantity is the potential enstrophy, the squared potential vorticity of the disturbance. The variational problem constructed with this metric will be referred to the "Q variational problem."

The idea of using potential enstrophy in a variational principle is not new, e.g., see Blumen (1968) or Bretherton and Haidvogel (1976). Potential enstrophy plays a central role in quasi-geostrophic dynamics, and it should be recalled that potential vorticity mixing, a useful concept in geostrophic turbulence, is closely related to this conserved quantity. Furthermore, when applying the energy method to quasi-geostrophic motions on a β-plane, the β effect disappears from the variational problem; a variational problem based on potential enstrophy, however, retains it.

Finally, one could try to build up a "hybrid" variational quantity, i.e., a variational problem with a metric consisting of a linear combination of the two previous metrics, exactly in the same spirit as Joseph's free parameter theory, with adjustment on this free parameter in order to obtain sharper results. We shall call this third approach the "λ variational problem."

These three approaches are developed below.

a. *The E variational problem*

After some algebra, an evolution equation for the total energy of a disturbance is readily obtained, in the form

$$\frac{dE}{dt} = -rD_E + P_E, \tag{4.1}$$

where

$$E = \tfrac{1}{2}\langle|\nabla\phi_1|^2 + |\nabla\phi_2|^2 + \mathrm{F}(\phi_1 - \phi_2)^2\rangle, \tag{4.2a}$$

$$D_E = \langle|\nabla\phi_1|^2 + |\nabla\phi_2|^2\rangle, \tag{4.2b}$$

$$P_E = \left\langle \tfrac{1}{2}\mathrm{F}V_S\left(\phi_1\frac{\partial\phi_2}{\partial x} - \phi_2\frac{\partial\phi_1}{\partial x}\right)\right\rangle, \tag{4.2c}$$

where the angle braces denote space averaging (in this case, over one wavelength in the x and y directions), del operators are two-dimensional, and ϕ_1, ϕ_2 are the disturbances in each layer. The notations E, D_E and P_E are completely similar to those of Section 3.

We are thus interested in the perturbations which maximize P_E/D_E; in terms of calculus of variations, we impose

$$\delta(D_E - \mu P_E) = 0, \tag{4.3}$$

where δ is the variation symbol, while μ is a Lagrange multiplier.

This relation yields an eigenvalue problem; for each eigenfunction (ϕ_1, ϕ_2), there will correspond an eigenvalue μ satisfying:

$$P_E/D_E = 1/\mu. \tag{4.4}$$

Euler-Lagrange equations are given by

$$\nabla^2\phi_1 - \frac{1}{2}\mu \mathrm{F} V_S \frac{\partial \phi_2}{\partial x} = 0, \tag{4.5a}$$

$$\nabla^2\phi_2 + \frac{1}{2}\mu \mathrm{F} V_S \frac{\partial \phi_1}{\partial x} = 0. \tag{4.5b}$$

This linear coupled system of equations is easy to solve and, taking into account the periodic behavior of the perturbations, we obtain

$$\mu = 2K^2/k\mathrm{F}V_S, \tag{4.6}$$

where $K^2 = k^2 + l^2$, and (k, l) is a wavevector.

In order to obtain information on the marginal stability curves, let k and l be fixed; the critical value of r beyond which the flow will be stable is given, by the argument developed in Section 3, as

$$r > \frac{1}{\mu} = \frac{k\mathrm{F}V_S}{2K^2}. \tag{4.7}$$

This relation can also be interpreted as follows: let r be fixed; for a perturbation characterized by wavenumber (k, l), stability is ensured if

$$V_S < \frac{2K^2 r}{\mathrm{F}k} \tag{4.8}$$

or

$$V_S < V_C(k, K^2),$$

with

$$\frac{V_C \mathrm{F}^{1/2}}{r} = \frac{2K^2}{k\mathrm{F}^{1/2}}, \tag{4.9}$$

which is to be compared with (2.4).

The curves defined by (4.9) represent marginal stability curves associated to our variational problem. We must keep in mind that only sufficient conditions for stability have been obtained. For a given wavenumber and a given shear, stability is ensured for values of the shear below these curves, but nothing can be said for shears above the limit given by (4.9). Stability as well as instability is possible.

The obtained stability curves are similar to the marginal stability curves of linear theory (see Fig. 1.); the latter always lies above the former and the point $K^2 = \mathrm{F}$ is common to both. Linear and nonlinear stability curves are very close in the region $0 < K^2 < \mathrm{F}$ but tend to separate for $K^2 > \mathrm{F}$, giving rise there to a completely different behavior.

b. The Q variational problem

We define Q as the sum of the squared potential vorticity of each layer (after integration in space), i.e.,

$$Q = ½\langle q_1^2 + q_2^2\rangle, \tag{4.10}$$

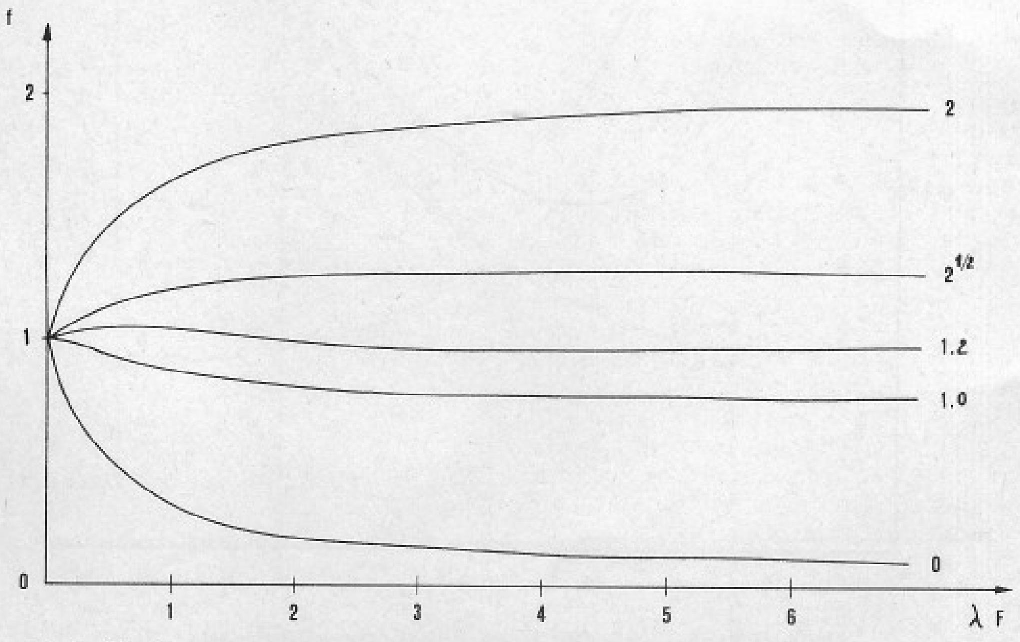

FIG. 2. The eigenvalue μ as a function of parameter λ for various values of K^2/F ($f = \mu^2 V_S^2 \mathrm{F}^2 k^2/4K^4$).

with

$$\left.\begin{aligned} q_1 &= \nabla^2\phi_1 - \mathrm{F}(\phi_1 - \phi_2) \\ q_2 &= \nabla^2\phi_2 + \mathrm{F}(\phi_1 - \phi_2) \end{aligned}\right\}.$$

Again, after some algebra, we obtain the relation

$$\frac{dQ}{dt} = -rD_Q + P_Q, \tag{4.11}$$

where

$$D_Q = \langle(\nabla^2\phi_1)^2 + (\nabla^2\phi_2)^2 + \mathrm{F}|\nabla(\phi_1 - \phi_2)|^2\rangle, \tag{4.12a}$$

$$P_Q = \left\langle \mathrm{F}^2 V_S\left(\phi_1\frac{\partial\phi_2}{\partial x} - \phi_2\frac{\partial\phi_1}{\partial x}\right)\right\rangle. \tag{4.12b}$$

Similar arguments as those developed in Sections 3 and 4a lead to the Euler-Lagrange equations

$$\nabla^4\phi_1 - \mathrm{F}\nabla^2(\phi_1 - \phi_2) - \mu\mathrm{F}V_S\frac{\partial\phi_2}{\partial x} = 0, \tag{4.13a}$$

$$\nabla^4\phi_2 + \mathrm{F}\nabla^2(\phi_1 - \phi_2) + \mu\mathrm{F}V_S\frac{\partial\phi_1}{\partial x} = 0. \tag{4.13b}$$

And, for fixed k and l, stability curves are defined by $V_S = V_C$ and

$$\frac{V_C\mathrm{F}^{1/2}}{r} = \left(\frac{K^2}{\mathrm{F}}\right)^{3/2}\frac{(K^2 + 2\mathrm{F})^{1/2}}{k}. \tag{4.14}$$

This relation must also be compared to (2.4).

The curves obtained in (4.14) are again represented in Fig. 1 for comparison with the marginal, linear stability curves. These curves intersect at $K^2 = \sqrt{2}\mathrm{F}$ and remain close to each other when $K^2 > \sqrt{2}\mathrm{F}$.

c. The λ variational problem

Given the two previous independent positive quantities, one is tempted to follow Joseph's method and combine these quantities into a single one, by use of a control parameter λ (which is positive). Then for each λ, a marginal stability curve will emerge and the best one (remember that the variational method leads to sufficient conditions of stability only) can be chosen.

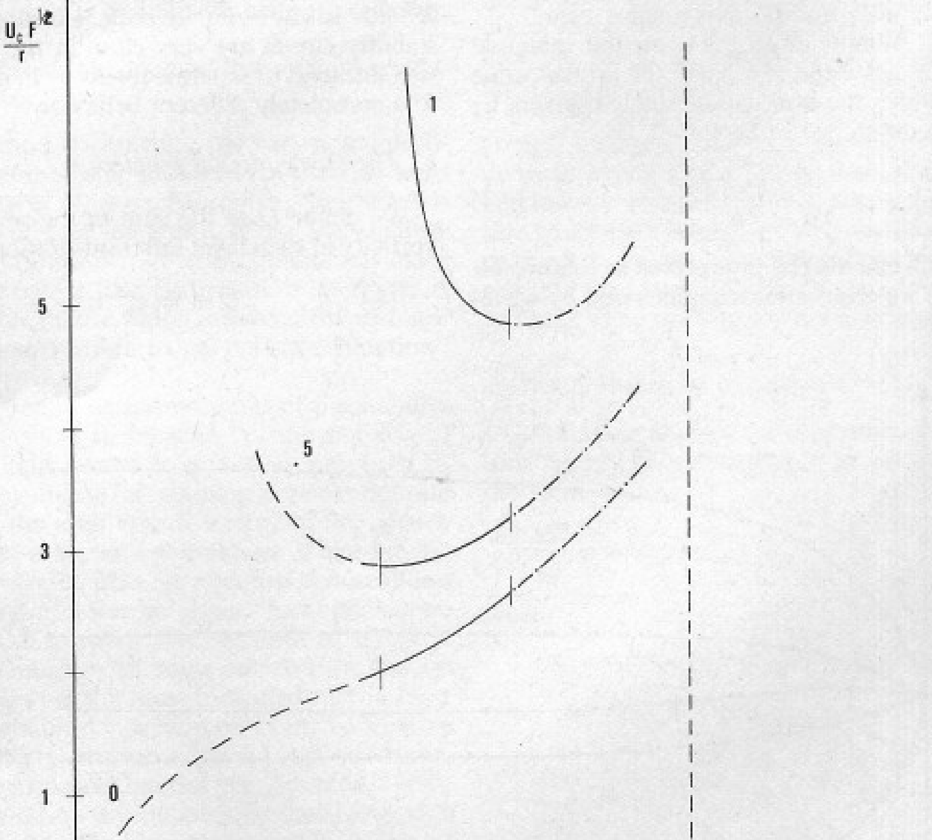

FIG. 3. Construction of the λ-variational problem stability curves.

Therefore, we pose

$$E_\lambda = E + \lambda Q, \quad \lambda \geq 0, \tag{4.15}$$

and from (4.11) and (4.1)

$$\frac{dE_\lambda}{dt} = -rD_\lambda + P_\lambda, \tag{4.16}$$

with

$$D_\lambda = D_E + \lambda D_Q,$$

$$P_\lambda = P_E + \lambda P_Q = (1 + 2\lambda \mathrm{F})P_E.$$

A new variational problem arises:

$$\delta(D_\lambda - \mu P_\lambda) = 0, \tag{4.17}$$

with the associated Euler-Lagrange equations

$$\lambda\nabla^4\phi_1 - \nabla^2\phi_1 - \lambda \mathrm{F}\nabla^2(\phi_1 - \phi_2) - \tfrac{1}{2}\mu(1 + 2\lambda\mathrm{F})\mathrm{F}V_S\frac{\partial\phi_2}{\partial x} = 0, \tag{4.18a}$$

$$\lambda\nabla^4\phi_2 - \nabla^2\phi_2 + \lambda \mathrm{F}\nabla^2(\phi_1 - \phi_2) + \tfrac{1}{2}\mu(1 + 2\lambda\mathrm{F})\mathrm{F}V_S\frac{\partial\phi_1}{\partial x} = 0. \tag{4.18b}$$

At this stage, it is worth noting that

$\lambda \to 0$ yields the E problem,

$\lambda \to +\infty$ yields the Q problem.

The resulting eigenvalue problem for μ gives

$$4K^4[\lambda(K^2 + 2\mathrm{F}) + 1](\lambda K^2 + 1) = \mu^2(1 + 2\lambda\mathrm{F})^2\mathrm{F}^2V_S{}^2k^2, \tag{4.19}$$

where

$$K^2 = k^2 + l^2.$$

So far, λ was a free positive parameter. We shall now

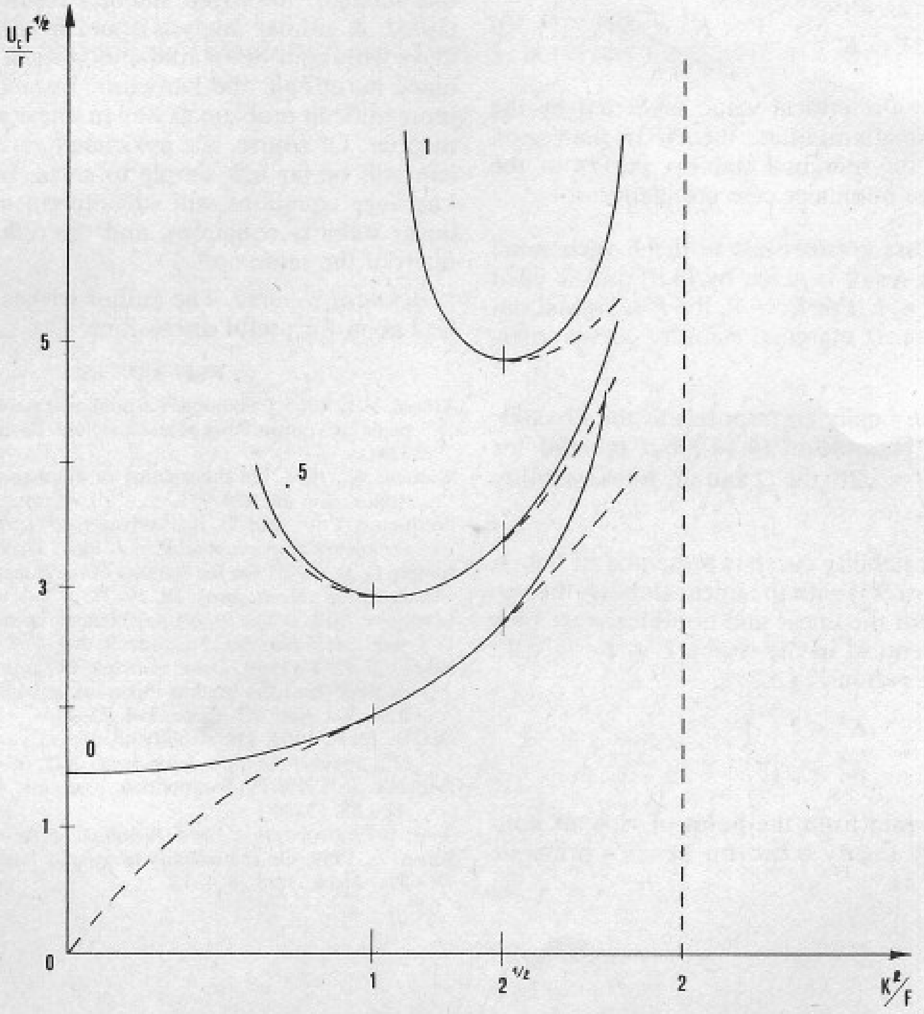

FIG. 4. Comparison between stability curves: solid line, linear theory; broken line, λ–variational problem.

select the best value of λ in order to reach the best limit of stability. This best limit corresponds of course to the largest value of μ considered as a function of λ. It is easy to verify that this maximum is achieved for (Fig. 2)

1) $\lambda = \dfrac{K^2 - F}{2F^2 - K^4}$ when $F < K^2 < \sqrt{2}F$

(the parameter λ must be positive),

2) $\lambda = 0$ when $K^2 \leq F$

3) $\lambda \to +\infty$ when $K^2 \geq \sqrt{2}F$.

It is by no means surprising that the best value for λ is a function of the perturbation wavenumber. We now consider the stability curves for each of these three cases. Stability is ensured for $V_S < V_C$ and as follows:

The first case corresponds to

$$\frac{V_C F^{1/2}}{r} = \frac{2KF^{1/2}}{k(2F - K^2)^{1/2}} \quad F < K^2 < \sqrt{2}F, \quad (4.20)$$

which is exactly the critical value predicted by the linear, infinitesimal amplitude theory! In the region defined above, the marginal stability curves in the linear and in the nonlinear case are identical;

The second case corresponds to the E-variational problem, which result is given by (4.9) and is valid for the range $K^2 \leq F$. For $K^2 = F$, the E-marginal stability and the linear marginal stability curves intersect;

The third case finally corresponds to the Q variational problem [expression (4.14)] but is valid for $K^2 \geq \sqrt{2}F$. At $K^2 = \sqrt{2}F$, the Q and the linear stability curves intersect.

The resulting stability curve is presented in Fig. 3, and Fig. 4 compares it with the linear stability theory. Stability limits for the linear and nonlinear case thus appear to be identical in the region $F \leq K^2 \leq \sqrt{2}F$; for the "outside regions", i.e.,

$$\left.\begin{array}{l} K^2 < F \\ K^2 > \sqrt{2}F \end{array}\right\},$$

nothing can be said from the point of view of nonlinear instability theory based on Serrin's principle and its extensions.

5. Conclusions

A stability investigation based on Serrin's principle of "stability in the mean," when applied to the study of dissipative flows of geophysical fluid dynamics (this method requires ultimately that the stability threshold be determined by dissipation) appears to be promising. The substituted variational problem, being linear, is far simpler to resolve than the fully nonlinear (e.g., Lyapunov, 1907) approaches and its introduction is most natural.

Moreover, this article shows that a pertinent choice of the maximizing problem is fruitful when based on more general positive quantities (i.e., not restricted to the perturbation kinetic energy).

Many geophysical problems contain such positive definite quantities, such as the total energy or the potential enstrophy, and these seem to be good candidates for playing the role of a metric characterizing the departure from the mean basic state.

Indeed the simple example treated shows how, by this method, improved stability results can be obtained. A similar analysis could be made for basic flows with both horizontal and vertical shears (combined barotropic and baroclinic instability), a much more difficult problem as well in linear and nonlinear theories. Of course, the associated variational problem will be far less simple to solve, but the Euler-Lagrange equations will still remain similar to the linear stability equations, and the difficulty will remain of the same order.

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